

Estimating Aggregate Production Function with I(2) Capital Stock¹

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Abstract

One of the most important features of estimating a production function is the presence of an I(2) capital stock series. Given the empirical regularity that the first difference of capital stock -i.e. investment- is an I(1) series, capital stock tends to have a double unit root. Using Ethiopian data from 1960/61 to 2001/02, we showed that the existence of cointegrating relationship is rejected under the I(1) analysis while the I(2) analysis fail to reject the existence of cointegrating relationship. This indicates the possibility of polynomial cointegration. We also argued that the polynomial cointegration can be motivated theoretical apart from being an empirical issue alone.

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1. Introduction

Specification and estimation of production function had been the workhouse of most economic theories. In a macro economics context, the growth literature *a la* the Solow model extensively employs aggregate production function and its parameters to come up with important conclusions. Among others, Mankiw et al (1992) used a Cobb-Douglas production function to test the implications of the Solow model while Easterly and Levin (2001) used it for their growth accounting analysis on the relative importance of total factor productivity *visa vise* total factor accumulation.

In cross country growth analysis, estimation of the Cobb-Douglas production function is fairly straight forward. Most of the empirical studies proxy capital by investment (scaled by GDP) and labor force by the economically active population and proceed to estimation either in cross section or panel framework. The problem of estimation, however, would crop-up when estimating a production function for a single country in the presence of stochastic trend in the time series variables. The problem would be serious as capital is a stock variable which invariably contain double unit root. In fact many applied macroeconomists in Africa do confront this problems when they attempt to formulated simple macro model for use in budget formulation and short run forecasting, say for use in the Medium Term Expenditure Framework (MTF) which is being widely employed in many Ministries of Finance Offices in Africa. This is at least our experience in Kenya and Ethiopia. The nature of the data in other African countries is not different and hence the problem could be found across countries in the continent (and elsewhere else). In this chapter we have attempted to address this applied problem.

In the presence of $I(2)$ variable, the usual approach is to difference the $I(2)$ series and employ the $I(1)$ analysis on the differenced and the other $I(1)$ variables. However, this restricts the possibility of having multi-cointegration relationships among the variables and hence imposes a priori restriction that there are only linear cointegrating relationships. Allowing for multi-cointegration or polynomial cointegration is a less restrictive description of the data that lets us treat different theoretically plausible relationships. This paper would, thus, employ an $I(2)$ framework to address the issue of $I(2)$ -ness in estimating aggregate production function. The method is illustrated using the Ethiopian data for the period 1960/61 to 2001/02.

2. The production function and data

Following the growth literature we considered a simple Cobb-Douglas production function⁴ given as in equation (2.1) and linearized in equation (2.2).

$$Y = AK^\alpha L^{1-\alpha} \quad 2.1$$

$$\ln Y = \beta + \alpha \ln K + 1 - \alpha \ln L \quad 2.2$$

Where Y is total output; A is technological progress, K is total physical capital, L is the total labor force and β is logarithm of A.

Estimating the production function in [2] using time series data requires addressing the stochastic trends in the variables. If all the variables are integrated of order one- I(1)-, estimation can be proceeded by testing for the existence of a common trend among the variables. When it is the case that one or more of the variables are integrated of higher order, our cointegration test in I(1) framework may give a misleading result.

Figure 1 presents the levels and growth rates of GDP, capital and labor series used for our analysis. As can be seen from the figure, the smooth trend of capital stock and its trending growth rate suggest its I(2)-ness. The smooth trend of labor force also suggests that it may be an I(2) series. The growth rate of the labor force, as shown in the figure, does not show a strong trend reverting behavior which also imply that the labor force series might border I(2)-ness. On the other hand the GDP series shows an I(1) behavior with its growth rate being trend reverting. The univariate unit root tests conducted using the Pantula principle could not also reject that capital is an I(2) series and output is I(1). The case for the labor force is not conclusive. The DF statistic shows that labor force is an I(1) series while the ADF statistics could not reject the I(2) null⁵.

At this juncture it may be necessary to say some about the nature of the data. We obtained GDP and gross capital formation from the national accounts compiled by Ministry of Finance and Economic Development. However, as frequent revision of the

⁴ It should be noted that the preference for a CD production function doesn't imply that it is the best though it is a workhorse in the Solow model and the growth literature in general. Practitioners can experiment with other practical functions such as the Constant Elasticity of Substitution (CES) and translog formulation.

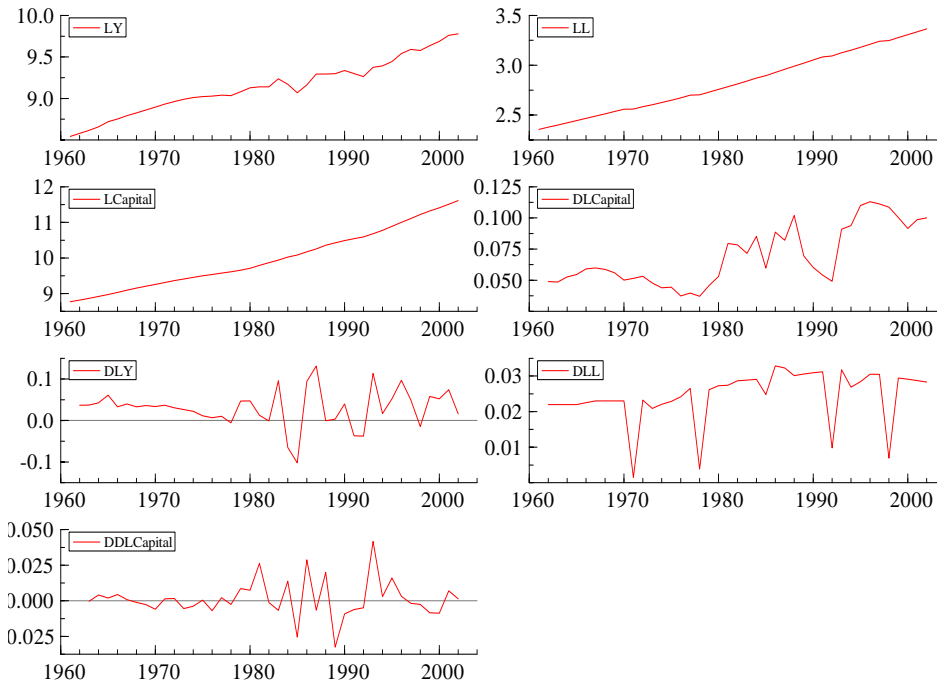
⁵ As one of the anonymous referees suggested, we also tested for trend stationarity and found that the de-trended series are nonstationary suggesting that the variables contain stochastic trend rather than deterministic one.

national accounts is prevalent, it is difficult to get reliable data that spans for the whole of the four decades. Specifically, the national accounts data from 1960/61 to 92/93 (referred as the old series) and from 1980/81 to 2000/01 (referred as the new series) are generated using different definitions. One way of addressing this problem of change in definitions of our dependent variable (GDP) is to introduce an impulse dummy in our regression. The alternative is to address the change in definition explicitly by looking at how it affected the series. Following the later method and comparing the old and new series of GDP reveals that the change in the definition seems to have a level effect leaving the growth rate somehow unaffected. In such a case we can fairly impose the growth rate of the old series on the new series and extrapolate the GDP series backward. We thus used the later procedure to adjust both the GDP and gross capital formation series of the national accounts.

Regarding capital stock, there is no estimate of the capital stock in the country. We generated the data for capital stock using the capital accumulation equation (where K_t is current capital stock and K_{t-1} lagged level of capital stock): $K_t = K_{t-1} + Investment - Depreciation$. As we don't also have the initial value of capital stock (K_{t-1}) we used an estimated incremental capital output ratio (ICRO) and generated K_{t-1} as: $K_{t-1} = ICRO * Output_{t-1}$.

The data for labor force is extracted from the World Bank's African Development Indicator CDROM (2005). This database compiles data from 1965 and hence we extrapolated the labor force series backward using exponential smoothing technique as our sample period goes back to 1960/61. The labor force series exhibited some unexpected trend. We could not explain the abrupt fall in the growth rate of labor force in 1970, 1977, 1993 and 1997. However, the retrenchment of labor force following the economic liberalization program and the HIV/AIDS epidemic could be plausible conjectures in explaining the sharp decline in the growth rate of labor force. The data for labor force is extracted from the World Bank's African Development Indicator CDROM (2005). This database compiles data from 1965 and hence we extrapolated the labor force series backward using exponential smoothing technique as our sample period goes back to 1960/61. The labor force series exhibited some unexpected trend. We could not explain the abrupt fall in the growth rate of labor force in 1970, 1977, 1993 and 1997. However, the retrenchment of labor force following the economic liberalization program and the HIV/AIDS epidemic could be plausible conjectures in explaining the sharp decline in the growth rate of labor force.

Figure 1



3. The econometric framework

The empirical analysis follows the Johansen (1995, 1997) multivariate cointegration framework based on a vector autoregressive process.

Consider the k^{th} order VAR(k) of the form

$$\mathbf{x}_t = A_1 \mathbf{x}_{t-1} + A_2 \mathbf{x}_{t-2} + \dots + A_p \mathbf{x}_{t-k} + \varepsilon_t \quad [3.1]$$

where \mathbf{x}_t is $p \times 1$ vector of endogenous variables, A_i is $p \times p$ matrix of parameters, and $\varepsilon_t \sim \text{IN}(0, \Sigma)$

Upon repeated re-parameterization, equation (3.1) can be written as

$$\Delta \mathbf{x}_t = \sum_{i=1}^{p-1} \Gamma \Delta \mathbf{x}_{t-i} + \Pi \mathbf{x}_{t-p} + \varepsilon_t \quad [3.2]$$

Where $\Gamma = -[I - \sum_{j=1}^i \mathbf{A}_j]$ and $\Pi = -[I - \sum_{i=1}^p \mathbf{A}_i]$

If the \mathbf{x}_t vector is cointegrating, the Π vector will have a reduced rank ($r < p$) so that it can be factorized in to α and β matrices, both of dimension $(p \times r)$, where r is the rank of π which is the same as the cointegration rank, such that $\pi = \alpha\beta'$. The matrix α is the matrix of weights with which each cointegrating vector enters the p equations of the VAR while the matrix β is a matrix of long run coefficients or cointegrating parameters. In addition to this, for the system to generate I(1) process, $\alpha'_{\perp} \Gamma \beta_{\perp}$, where α_{\perp} and β_{\perp} are orthogonal complement matrices of α and β , (so that $\alpha'_{\perp} \alpha = \beta'_{\perp} \beta = 0$) needs to have a full rank (i.e., it has to be stationary)⁶. Otherwise, the system would be integrated of higher order. Thus, to account for I(2) system we need a reduced rank restriction on $\alpha'_{\perp} \Gamma \beta_{\perp}$.

Following Johansen (1997) the I(2) model can be represented as

$$\Delta^2 X_t = \Pi X_{t-1} - \Gamma \Delta X_{t-1} + \sum_{i=1}^{k-2} \Psi_i \Delta^2 X_{t-1} + \varepsilon_t \quad [3.3]$$

Where ε_t is iid and $\Psi_i = -\sum_{j=i+1}^{k-1} \Gamma_j$

As in the I(1) model assuming that the Π matrix has a reduced rank; and allowing the $\alpha'_{\perp} \Gamma \beta_{\perp}$ to have a reduced rank to accommodate higher order of integration, we can analyze the I(2) system. With $\alpha'_{\perp} \Gamma \beta_{\perp}$ having a reduced rank, it can be parameterized into ξ and η matrices. Thus, the joint pair of reduced rank conditions of the I(2) model can be stated as

$$\Pi = \alpha\beta' \quad \alpha, \beta (p \times r), r < p \quad [3.4]$$

⁶ Johanson (1988, 1991, 1995) has shown that the cointegrated VAR model can also be given by an alternative 'common stochastic trend' representation $X_t = C \sum_{i=1}^t \varepsilon_i + C(L)\varepsilon_t$ where $C = \beta_{\perp}(\alpha'_{\perp} \Gamma \beta_{\perp})'$

¹ α'_{\perp} ; β is $p \times (p-r)$ dimension; $\beta'_{\perp} \beta = 0$ and $\text{rank}(\beta_{\perp} \beta) = p - [C(L)]$ such that $C(L)\varepsilon_t$ corresponds to a p -dimensional I(0) component. It can be shown that although X_t is p -dimensional, the vector series is driven by just $p-r$ common stochastic I(1) trend $\sum_{i=1}^t \varepsilon_i \alpha'_{\perp}$ (see Haldrup, 1998:627)

$$\alpha'_{\perp} \Gamma \beta_{\perp} = \xi \eta' \quad \xi, \eta \ (p-r) \times s, \ s < (p-r)$$

Johansen (1995, 1997) showed that the space spanned by the system can be decomposed into I(0), I(1) and I(2) trends. Three mutually orthogonal matrices β , β_1 and β_2 , where

$$\beta_1 = \beta_{\perp} (\beta'_{\perp} \beta_{\perp})^{-1} \eta \quad \text{and} \quad \beta_2 = \beta_{\perp} \eta_{\perp}$$

divide the p-dimensional space into different cointegration relationships. The process can be described as:

$$\begin{array}{ll} \beta_2' X_t & p-r-s \text{ dimensional I(2) trend (where } s < (p-r)) \\ (\beta, \beta_1)' X_t & \text{I(1) relations that reduce the integration order from 2} \\ & \text{to 1.} \\ \beta' X_t - \delta \beta_2' \Delta X_t & \text{I(0) polynomial cointegration result where} \\ \delta = \bar{a}' \Gamma \bar{\beta}_2 & \text{of dimension } r \times (p-r-s) \end{array}$$

The number of polynomial cointegrating relations cannot exceed the number of I(2) trends requiring that $r \geq p-r-s$.

The estimation procedure follows Johansen's (1995) two step procedure. The first step involves solving the reduced rank problem for the Π matrix for each value of $r = 0, \dots, p-1$. On the second stage, the estimated α and β will be used to solve the reduced rank regression problem associated with $\alpha'_{\perp} \Gamma \beta_{\perp} = \xi \eta'$ which will be solved for $s = 0, 1, \dots, p-r-1$. At each step the trace statistics will be generated. On the first step, the trace statistics Q_r for a rank of Π equal r against the unrestricted rank p is computed. On the second step, the trace statistics conditional on the first step gives $Q_{r,s}$ for a rank of $\alpha'_{\perp} \Gamma \beta_{\perp}$ equal s against the unrestricted rank $p-r$. The sum of the two tests, $[S_{r,s}]$ gives a simultaneous test for the model $[H_{r,s}]$ against the unrestricted alternative. The model $[H_{r,s}]$ is rejected if $[H_{i,j}]$ is rejected for all for all $i < r$ and $j \leq s$.

4. Empirical results

The cointegration analysis is preceded by determining the data congruency of the VAR. Using the data from 1960/61 – 2001/02, VAR (2) appears to be a valid specification based on the Akaike Information Criterion. The restriction that ‘lags higher than 2 are all zero’ cannot also be rejected using F-statistics confirming the VAR (2) specification.

We first conducted an I(1) cointegration analysis using our VAR (2) specification. The result shows that output, capital and labor are not cointegrated implying that the variables do not have a stable relationship that leads to an I(0) relationship. That is, the I(2) variables might cointegrate as C(2,1) but they do not cointegrate with the I(1) variable leading to an I(1) residual. We proceed to an I(2) analysis to test for a possible polynomial cointegration.

Table 4.1: Testing cointegration in I(1) framework

		Trace test	[Prob]
H ₀ : Ranks	0	26.721	[0.111]
	1	10.987	[0.216]
	<u>2</u>	0.43605	[0.509]

To identify the rank indices of the I(2) model, r and s , we used the trace statistics test, $S_{r,s}$ reported in Table 4.2. The test starts from the most restrictive model, $H_{0,0}$ in the upper left hand side and proceeds down the columns, from top left to bottom right, stopping at the first acceptance *a la* Pantula principle.

Table 4.2: I(2) test for the rank order

r		$S_{r,s}$		Q_r
0	92.308 <i>(0.0003)</i>	58.883 <i>(0.0229)</i>	44.007 <i>(0.0404)</i>	44.005 <i>(0.0036)</i>
1		36.813 <i>(0.0495)</i>	18.363 0.3405	18.341 <i>(0.0897)</i>
2			6.5354 <i>(0.4273)</i>	6.5152 <i>(0.1594)</i>
$p-r-s$	3	2	1	0

† The (italics) figures are probabilities associated with the trace statistics

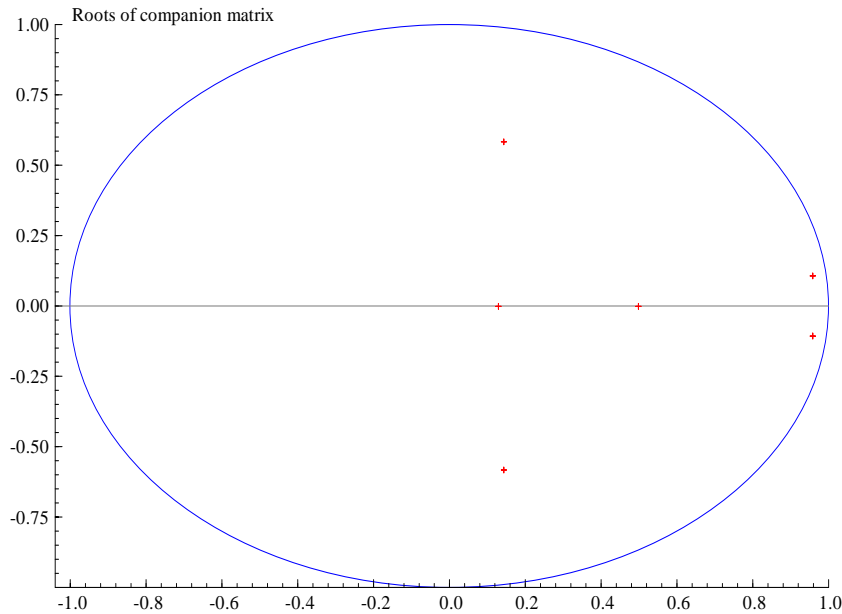
Our result shows that our first acceptance is when the number of I(2) trends in the system, $p-r-s=1$ and $r=1$. Contrary to our I(1) analysis, we obtained one stable cointegrating relationship with one I(2) trend. The lack of cointegration in our I(1)

analysis is visa vise the presence of it in the I(2) analysis may suggest the presence of polynomial cointegration. That is the I(2) variables may cointegrate as C(2,1) which would further cointegrate with the change in the I(2) variable and the remaining I(1) variables. As the number of polynomial cointegration cannot exceed the number of I(2) trends = 1, the cointegrating relationship that we obtained would be due to the polynomial term in our I(2) cointegration analysis.

We first adopted the usual approach that tests cointegration among the differenced I(2) variable and other I(1) variables. We used VAR (2) specification and tested cointegration among output, labor and change in capital (investment). Our result shows that the null hypothesis of zero cointegration relationship is rejected in favor of one cointegration relationship. The moduli of the companion matrix are all within the unit circle as shown in the figure below. However, the existence of large roots close to 1 may indicate that there may be some unaccounted trend close to I(2).

Table 4.3:

	Trace Statistics		Prob
	0	65.674	[0.000] **
	1	21.302	[0.169]
$H_0: Rank \leq$	2	8.2542	[0.238]



Given the problem of unaccounted I(2) trend and to allow for other type of relationship, we tried to identify the polynomial cointegration relationship. We tried to motivate the polynomial cointegration that the data exhibited theoretically. That is, the polynomial cointegration may indicate that output depends on the common trend of capital and labor (for instance capital labor ratio) and the new investment made. The I(2) capital stock may cointegrate with the near I(2) labor force- C(2,1) which may further cointegrate with the change in the I(2) variables- capital. Change in capital stock can be a theoretically valid candidate in forcing this relationship to cointegrate and hence it can be used as the polynomial cointegrating term.

To test the polynomial cointegration, we created a variable that captures the common trend of capital and labor as capital labor ratio - i.e. $\ln K - \ln L$. Using this variable along with output and change in capital stock, we run a cointegration test in I(1) framework and obtained one cointegrating relationship with the roots of the companion matrix being all within a unit circle. This is an interesting result in supporting the polynomial cointegration relationship. It also underscores the importance of I(2) analysis in motivating different theoretical relationships. Table 4.4 reports this result.

Once the cointegration relationship is established, the analysis can proceed using the I(1) framework. The long run relationships can be identified using the β vector with the adjustment coefficients contained in the α vector discussed in the previous section. Hypothesis testing can also be conducted in the I(1) framework using the LR test.

Table 4.4

		Trace Statistics	Prob
	0	63.990	[0.000] **
	1	25.398	[0.055]
$H_0: Rank \leq$	2	8.3336	[0.232]

At this juncture, it is important to notice that the I(2) analysis is an important framework in detecting a polynomial cointegration relationship though the final analysis is based on an I(1) approach. As opposed to adopting the I(1) framework with the differenced I(2) variables, this approach is more flexible in allowing different types of relationships which are filtered out in the earlier approach.

5. Conclusion

We estimated a simple production function allowing for the presence of $I(2)$ variables. We showed that, in the presence of $I(2)$ capital stock for instance, the $I(1)$ cointegration analysis rejected the presence of cointegration among the variables. However, the $I(2)$ analysis showed the presence of one $I(2)$ trend with a possible cointegration as the rank $r=1$. This can be attributed to the polynomial cointegration allowed in the $I(2)$ analysis.

We also showed that the polynomial cointegration found in the data could also be motivated theoretically. Capital and labor may cointegrate as $C(2,1)$ forming an $I(1)$ capital labor ratio which along with investment (i.e. change in capital stock which is also the polynomial integration term) may give a stable relationship. This shows the importance of putting the problem in an $I(2)$ framework to address the higher order integration and their relationship with the lower integration order.

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